МЕХАНИКА ЖИДКОСТИ, ГАЗА И ПЛАЗМЫ

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ON SOLUTIONS OF A REDUCED MODEL FOR THE DYNAMICAL EVOLUTION OF CONTACT LINES

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Purpose: The goal of this study is to solve the linear advection--diffusion equation with a variable speed on a semi-infinite line. The variable speed is determined by an additional condition at the boundary, which models the dynamics of a contact line of a hydrodynamic flow at 180˚contact angle.
Approach: The investigation is carried out by an application of Laplace transform in spatial coordinate. Properties of Green's function for the fourth-order diffusion equation are used in analysis of implicit solutions of the linear advection-diffusion equation.
Findings: We prove local existence of solutions of the initial-value problem associated with the set of over-determining boundary conditions in the form of the fractional power series in time variable. We also analyze the explicit solutions in the case of a constant speed to show that the inhomogeneous boundary condition induces change of convexity of the flow at the contact line in a finite time.

Key words: linear advection-diffusion equation, variable speed, contact line, Laplace transform, Green’s function.

1. Introduction

Contact lines are defined by the intersection of the rigid and free boundaries of the flow. Flows with the contact line at 180˚ contact angle were discussed in [2, 4], where corresponding solutions of the Navier-Stokes equations were shown to have no physical meanings. Recently, a different approach based on the lubrication approximation and thin film equations was developed by Benilov & Vynnycky [1].

As a particularly simple model for the flow shown on Fig. 1, the authors of [1] derived the linear advection–diffusion equation for the free boundary $h(x,t)$ of the flow:

$$\frac{\partial h}{\partial t}+\frac{\partial^4 h}{\partial x^4}=V(t)\frac{\partial h}{\partial x}, \quad x>0, \quad t>0. \tag{1}$$

The contact line is fixed at $x = 0$ in the reference frame moving with the velocity $-V(t)$ and is defined by the boundary conditions $h(0,t) = 1$ and $h_x(0,t) = 0$. The flux conservation is expressed by the boundary condition $h_{xxx}(0,t)=-\frac{1}{2}$ (set $\alpha^3 = 3$ in equations (5.12)–(5.13) in [1]).

We assume that h, h_x, $h_{xx} \to 0$ as $x\to\infty$: in fact, any constant value of h at infinity is allowed thanks to the invariance of the linear advection–diffusion equation (1) with respect to the shift and scaling transformations. With three boundary conditions at $x = 0$ and the decay conditions as $x \to \infty$, the initial-value problem for equation (1) is over-determined and the third (over-determining) boundary condition at $x = 0$ is used to find the dependence of V on t.

We shall consider the initial-value problem with the initial data $h(x,0) = h_0(x)$ for a suitable function h_0. In particular, we assume that the profile $h_0(x)$ decays monotonically to zero as $x\to\infty$

and that 0 is a non-degenerate maximum of h_0 such that $h_0(0) = 1$, $h_0'(0) = 0$, and $h_0''(0) < 0$, see Fig. 1. If the solution $h(x,t)$ losses monotonicity in x during the dynamical evolution, for instance, due to the value of $h_{xx}(0,t)$ crossing 0 from the negative side, then we say that the flow becomes non-physical for further times and the model breaks. Simultaneously, this may mean that the velocity $V(t)$ blows up, as it is defined for sufficiently strong solutions of the advection–diffusion equation (1) by the pointwise equation:

$$h_{xxxxx}(0,t) = V(t)\, h_{xx}(0,t), \tag{2}$$

which follows by differentiation of (1) in x and setting $x \to 0$.

The main claim of [1] based on numerical computations of the reduced equation (1) as well as more complicated thin-film equations is that for any suitable h_0, there is a finite positive time t_0 such that $V(t) \to -\infty$ and $h_{xx}(0,t) \to 0^-$ as $t \uparrow t_0$. Moreover, it is claimed that $V(t)$ behaves near the blowup time as the logarithmic function of t, e.g.

$$V(t) \sim C_1 \log(t_0 - t) + C_2 \quad \text{as} \quad t \uparrow t_0, \tag{3}$$

where C_1, C_2 are positive constants.

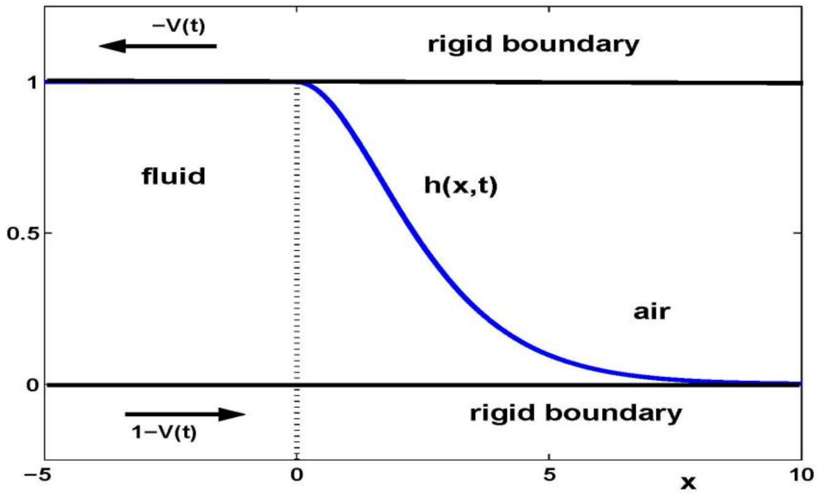

Fig. 1. Schematic picture of the flow between rigid boundaries

This paper is devoted to analytical studies of solutions of the advection–diffusion equation (1) and the effects coming from the inhomogeneous boundary condition $h_{xxx}(0,t) = -\frac{1}{2}$ associated with the flux conservation. In particular, we rewrite the evolution equation for the variable $u = h_x$ in the form

$$u_t + u_{xxxx} = V(t)\, u_x, \quad x > 0, \quad t > 0, \tag{4}$$

subject to the boundary conditions at the contact line

$$u(0, t) = 0, \quad u_{xx}(0,t) = -\frac{1}{2}, \quad u_{xxx}(0,t) = 0, \quad t \geq 0, \tag{5}$$

where the boundary conditions $u_{xxx}(0,t) = h_{xxxx}(0,t) = 0$ follows from the boundary conditions $h(0,t) = 1$ and $h_x(0,t) = 0$ as well as the advection-diffusion equation (1) as $x \to 0$.

To simplify the problem, we shall also consider the model for given constant $V(t) = V_0$ and drop the third over-determining boundary conditions at the contact line:

$$\begin{cases} u_t + u_{xxxx} = V_0 u_x, & x > 0, t > 0, \\ u(0,t) = 0, u_{xx}(0,t) = -\dfrac{1}{2}, & t \geq 0. \end{cases} \tag{6}$$

Both problems (4)–(5) and (6) are considered under the initial condition $u(x,0) = u_0(x)$ with $u_0(0) = 0$, $u_0'(0) < 0$,and $u_0''(0) = -\frac{1}{2}$ as well as the decay condition u, u_x, $u_{xx} \to 0$ as $x \to \infty$.

Using Laplace transform in spatial coordinate and Green's function for the fourth-order diffusion equation, we derive an explicit solution of the boundary-value problem (6). In the case $V_0 = 0$, we show that the inhomogeneous boundary condition $h_{xxx}(0,t) = u_{xx}(0,t) = -\frac{1}{2}$ leads to the secular growth of the boundary value $h_{xx}(0,t) = u_x(0,t)$ to positive infinity as $t \to \infty$. As a result, even if $h_{xx}(0,t) < 0$ initially, the convexity of the solution $h(x,t)$ at the boundary $x = 0$ is lost in a finite time. In the case $V_0 < 0$, we show that no secular growth is observed but the convexity of the solution at the boundary is still lost in a finite time. Applying the same method, we prove local existence of solutions of the original boundary-value problem (4)–(5) in the form of fractional power series in time variable t. This prepares us to tackle the original conjecture on the finite-time blow-up in the dynamical evolution of the reduced model (1), which is still left opened for forthcoming studies.

The remainder of this paper is organized as follows. Section 2 reports explicit solutions of the boundary–value problem (6) for $V_0 = 0$ and $V_0 \neq 0$. Section 3 gives the local existence result for the boundary-value problem (4)–(5). Appendix A reviews properties of Green's function for the fourth-order diffusion equation.

2. Solution for $V(t) = V_0$

Because the coefficient $V(t)$ changes in time variable t in the framework of the original advection-diffusion equation (1), the Laplace transform in time t is not a useful method for this problem. On the other hand, the boundary-value problem (1) is formulated on half-line, and hence we can use Laplace transform in space variable x:

$$U(p,t) = \int_0^\infty e^{-px} u(x,t)\,dx, \qquad p > 0. \tag{7}$$

We shall develop this method to solve the boundary–value problem (6). The explicit solution of this problem will help us to analyze the effects of the inhomogeneous boundary condition $u_{xx}(0,t) = -\frac{1}{2}$ and the constant advection term $V(t) = V_0$ on the temporal dynamics of the advection-diffusion equation with the fourth-order diffusion.

Let us denote the boundary values:

$$\beta(t) = u_x(0,t), \quad \gamma(t) = u_{xxx}(0,t). \tag{8}$$

Using Laplace transform (7), we rewrite an evolution problem associated with the advection–diffusion equation (6):

$$\begin{cases} U_t + p^4 u - V_0 p U = \gamma(t) - \dfrac{1}{2} p + p^2 \beta(t), & t > 0, \\ U(p,0) = U_0(p), \end{cases} \tag{9}$$

where $U_0(p)$ is the Laplace transform of $u_0(x) = u(x,0)$. By using the variation of parameters, we obtain

$$U(p,t)=U_0(p)e^{-tp^4+tV_0p}+\int_0^t e^{-(t-s)p^4+(t-s)V_0p}(\gamma(s)-\frac{1}{2}p+p^2\beta(s))ds. \tag{10}$$

Using the inverse Laplace transform, we write this solution in the form:

$$\begin{aligned}u(x,t)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px-tp^4+tV_0p}(\int_0^\infty e^{-py}u_0(y)dy)dp\\&+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}\left(\int_0^t e^{-(t-s)p^4+(t-s)V_0p}\left(\gamma(s)-\frac{1}{2}p+p^2\beta(s)\right)ds\right)dp,\end{aligned} \tag{11}$$

where $Re(c)>0$ so that the singularities of the integrand in the complex p-plane remain to the left of the contour of integration.

If $t>0$ is finite, $u_0\in L^1(R_+)$, and $\beta,\gamma\in L^\infty_{loc}(R_+)$, Fubini's Theorem implies that the integration in p and in y, s can be interchanged. Let us introduce Green's function $G_t(x)$ for the fourth-order diffusion equation (see Appendix A):

$$G_t(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px-tp^4}dp=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-tk^4+ikx}dk=\frac{1}{\pi}\int_0^\infty e^{-tk^4}\cos(kx)dk.$$

Using Green's function, we can rewrite the solution (11) in the implicit form:

$$\begin{aligned}u(x,t)&=\int_0^\infty G_t(x+V_0t-y)u_0(y)dy-\frac{1}{2}\int_0^t G'_{t-s}(x+V_0(t-s))ds\\&+\int_0^t\left[G_{t-s}(x+V_0(t-s))\gamma(s)+G''_{t-s}(x+V_0(t-s))\beta(s)\right]ds.\end{aligned} \tag{12}$$

The solution is said to be in the implicit form, because the functions $\beta(t)$ and $\gamma(t)$ determined by the boundary conditions (8) are not specified yet.

We verify that $\lim_{x\to\infty}u(x,t)=0$, no matter what β and γ are, as long as they are bounded function of t. Indeed, by the Lebesgue's Dominated Convergence Theorem, we have

$$\int_0^\infty G_t(x+V_0t-y)u_0(y)dy\to 0 \quad \text{as} \quad x\to\infty$$

if $u_0\in L^1(R_+)$, because $G_t(x)\to 0$ as $x\to\infty$. On the other hand, the other three convolution integrals are bounded if $\beta,\gamma\in L^\infty_{loc}(R_+)$ and $t>0$ is finite, because G_t, G'_t and G''_t have integrable singularities at $t=0$. By the same Lebesgue's Dominated Convergence Theorem, these three integrals decay to zero as $x\to\infty$.

It follows from this construction that the only way to determine the functions $\beta(t)$ and $\gamma(t)$ in the solution (12) is to use the boundary conditions at $x=0$, e.g. the boundary conditions $u(0,t)=0$ and $u_x(0,t)=\beta(t)$. In what follows, this step is performed separately for the cases of $V_0=0$ and $V_0\neq 0$.

2.1. Case $V_0 = 0$

We rewrite the solution (12) for $V_0=0$:

$$u(x,t)=\int_0^\infty G_t(x-y)u_0(y)dy-\frac{1}{2}\int_0^t G'_{t-s}(x)ds+\int_o^t\left[G_{t-s}(x)\gamma(s)+G''_{t-s}(x)\beta(s)\right]ds. \tag{13}$$

Using (A.3) and (A.4) for Green's function $G_t(x)$ and the boundary condition $u(0,t)=0$, we evaluate this expression at $x=0$ and obtain an integral equation for β and γ:

$$-\frac{1}{4\pi}\Gamma\left(\frac{1}{4}\right)\int_0^t \frac{\gamma(s)}{(t-s)^{1/4}}ds+\frac{1}{4\pi}\Gamma\left(\frac{3}{4}\right)\int_0^t \frac{\beta(s)}{(t-s)^{3/4}}ds=\int_0^\infty G_t(-y)u_0(y)dy. \quad (14)$$

To use the boundary condition $u_x(0,t) = \beta(t)$, we shall recall from equation (A.5) that the function $G_t'''(x)$ behaves like $O(t^{-1})$ for any $x > 0$ and hence is not integrable in t at $t = 0$. Therefore, we have to be careful to differentiate the solution in the above convolution form. The last term of the solution (13) can be computed by using the Fourier transform:

$$v(x,t) := \int_0^t G_{t-s}''(x)\beta(s)ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(ik)^2 e^{ikx}\left(\int_0^t e^{-k^4(t-s)}\beta(s)ds\right)dk.$$

Differentiating this expression in x and integrating by parts in s, we obtain

$$\begin{aligned} v_x(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(ik)^3 e^{ikx}\left(\int_0^t e^{-k^4(t-s)}\beta(s)ds\right)dp = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{ikx}}{k}\left(\int_0^t \frac{d}{ds}\left(e^{-k^4(t-s)}\right)\beta(s)ds\right)dp \\ &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{ikx}}{k}\left(\beta(t)-\beta(0)e^{-k^4 t}-\int_0^t e^{-k^4(t-s)}\beta'(s)ds\right)dp \\ &= \frac{1}{2}\beta(t)-\beta(0)H_t(x)-\int_0^t H_{t-s}(x)\beta'(s)ds, \end{aligned} \quad (15)$$

where

$$H_t(x) := \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{-tk^4+ikx}}{k}dk = \frac{1}{\pi}\int_0^\infty \frac{e^{-tk^4}\sin(kx)}{k}dk = \int_0^x G_t(y)dy. \quad (16)$$

Here we note that all integrals are evaluated in the principal value sense, because the half-residue at $k = 0$ is canceled out in the resulting expression (15). Also we note that the decay of $v_x(x,t)$ to zero as $x \to \infty$ is satisfied because of the symmetry and normalization of G_t in (A.6). We can now use the boundary condition $u_x(0,t) = \beta(t)$ to obtain the exact value for $\beta(t)$:

$$\beta(t) = 2\int_0^\infty G_t'(-y)u_0(y)dy - \int_0^t G_{t-s}''(0)ds = 2\int_0^\infty G_t'(-y)u_0(y)dy + \frac{\Gamma(3/4)}{\pi}t^{1/4}. \quad (17)$$

After $\beta(t)$ is found uniquely from (17), $\gamma(t)$ is found uniquely from the integral equation (14). This computation completes the construction of the exact solution of the boundary-value problem (6) for $V_0 = 0$ (see also [5] for other solutions of this fourth-order diffusion equation). Now we turn to the analysis of the solution thus obtained.

Theorem 1. *Consider the advection-diffusion equation (6) for $V_0 = 0$ with the initial data $u_0 \in L^1(R_+)$. Then, there exists a solution $u \in L^\infty(R_+ \times R_+)$ of the evolution problem in the explicit form (13), where $\beta, \gamma \in L^\infty_{loc}(R_+)$ are defined by (14) and (17) and $\lim_{t\to\infty}\beta(t) = +\infty$.*

Proof. The convolution integral in the explicit expression (17) can be analyzed from the representation (A.5) for Green's function G_t. If $u_0 \in L^1(R_+)$, then

$$\left|\int_0^\infty G_t'(-y)u_0(y)dy\right| \le \frac{\|g'\|_{L^\infty}\|u_0\|_{L^1}}{t^{1/2}}, \qquad t > 0.$$

Therefore, $\beta \in L^\infty_{loc}(R_+)$ and $\beta(t) \sim t^{1/4}$ as $t \to \infty$ due to the second term in (17). Now, the integral equation (14) for γ(t) with a weakly singular kernel is well defined and solutions exist with $\gamma \in L^\infty_{loc}(R_+)$. Similarly, the solution $u \in L^\infty(R_+ \times R_+)$ is well defined by (13).

Remark 1 *One can show that there is no singularity of the solution for $\beta(t)$ as $t\to 0$ so that* $\beta(0)=u_0'(0)$ *by continuity. Also, one can show that the solution of the integral equation (14) for $\gamma(t)$ exists in the closed form:* $\gamma(t)=2\int_0^\infty G_t(-y)u_0'''(y)dy$.

Coming back to the original question, if $u_0(0) = 0$, $u_0'(0)<0$, and $u_0''(0)=-\frac{1}{2}$, then there is a finite value of $t_0\in(0,\infty)$ such that $u_x(0,t) > 0$ for all $t > t_0$, that is, $h(x,t)$ loses monotonicity at the boundary $x = 0$ in a finite time t_0 (recall that $u = h_x$). This dynamical phenomenon occurs because of the inhomogeneous boundary conditions $u_{xx}(0,t)=-\frac{1}{2}$ even in the absence of the advection term in the fourth-order diffusion equation (6).

2.2. Case $V_0\neq 0$

We have the solution in the implicit form (12) and we need to derive integral equations on the unknown functions $\beta(t)$ and $\gamma(t)$. One integral equation follows again from the boundary condition $u(0,t) = 0$:

$$-\int_0^t\left[G_{t-s}(V_0(t-s))\gamma(s)+G_{t-s}''(V_0(t-s))\beta(s)\right]ds=\int_0^\infty G_t(V_0t-y)u_0(y)dy-\frac{1}{2}\int_0^t G_{t-s}'(V_0(t-s))ds.\quad(18)$$

To find another integral equation from the boundary condition $u_x(0,t) = \beta(t)$, we have to use the technique explained in Section 2.1 and to compute the derivative of the solution (12) in x:

$$\begin{aligned}u_x(x,t)=&\int_0^\infty G_t'(x+V_0t-y)u_0(y)dy-\frac{1}{2}\int_0^t G_{t-s}''(x+V_0(t-s))ds\\&+\int_0^t G_{t-s}'(x+V_0(t-s))\gamma(s)ds+\frac{1}{2}\beta(t)-\beta(0)H_t(x+V_0t)\\&-\int_0^t H_{t-s}(x+V_0(t-s))\beta'(s)ds+V_0\int_0^t G_{t-s}(x+V_0(t-s))\beta(s)ds.\end{aligned}\quad(19)$$

We can now use the boundary condition $u_x(0,t) = \beta(t)$ to obtain another integral equation for β and γ:

$$\begin{aligned}&\beta(t)+2\beta(0)H_t(V_0t)+2\int_0^t H_{t-s}(V_0(t-s))\beta'(s)ds-2V_0\int_0^t G_{t-s}(V_0(t-s))\beta(s)ds\\&-2\int_0^t G_{t-s}'(V_0(t-s))\gamma(s)ds=2\int_0^\infty G_t'(V_0t-y)u_0(y)dy-\int_0^t G_{t-s}''(V_0(t-s))ds.\end{aligned}\quad(20)$$

The system of integral equations (18) and (20) completes the solution (12) for the case $V_0\neq 0$. Because of the original motivation to study behavior of the flow on Fig. 1 for large negative $V(t)$, see equation (3), we shall analyze the obtained solution for $V_0 < 0$ only.

Theorem 2. *Consider the advection-diffusion equation (6) for $V_0 < 0$ with the initial data* $u_0\in L^1(R_+)$. *Then, there exists a solution* $u\in L^\infty(R_+\times R_+)$ *of the evolution problem in the explicit form (12), where* $\beta,\gamma\in L^\infty(R_+)$ *are defined by (18) and (20) with*

$$\lim_{t\to\infty}\beta(t)=\frac{1}{2|V_0|^{1/3}} \qquad \lim_{t\to\infty}\gamma(t)=\frac{|V_0 t|^{1/3}}{2}. \tag{21}$$

Proof. Similarly to the proof of Theorem 1, it is easy to show from the integral equations (18) and (20) that if $u_0 \in L^1(R_+)$, then $\beta, \beta', \gamma \in L^\infty_{loc}(R_+)$. We shall now compute the limit of β(*t*) and γ(*t*) as $t \to \infty$:

$$\beta_\infty := \lim_{t\to\infty}\beta(t), \qquad \gamma_\infty := \lim_{t\to\infty}\gamma(t). \tag{22}$$

To deal with the first integral equation (18), we first notice the explicit computation by using the Fourier transform:

$$f(t) := \int_0^t G'_{t-s}(V_0(t-s))ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(ik)\left(\int_0^t e^{-s(k^4-ikV_0)}ds\right)dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{i(1-e^{-t(k^4-ikV_0)})}{k^3-iV_0}dk,$$

where the integrals in *s* and *k* can be interchanged by Fubini's Theorem and the integration is performed in the principal value sense. We can now explicitly compute the limit as $t \to \infty$ by using Lebesgue's Dominated Convergence Theorem:

$$\lim_{t\to\infty} f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{i}{k^3-iV_0}dk = \frac{-V_0}{\pi}\int_0^\infty\frac{dk}{k^6+V_0^2} = \frac{1}{3|V_0|^{2/3}}.$$

This computation gives the last term of the integral equation (18) as $t \to \infty$. To deal with the first term on the right-hand side of (18), we write

$$\int_0^\infty G_t(V_0 t - y)u_0(y)dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^\infty e^{-t(k^4-ikV_0)-iky}u_0(y)dy\right)dk = \int_{-\infty}^{\infty}e^{-t(k^4-ikV_0)}\hat{u}_0(k)dk,$$

where

$$\hat{u}_0(k) := \frac{1}{2\pi}\int_0^\infty e^{-iky}u_0(y)dy.$$

By Lebesgue's Dominated Convergence Theorem, this integral converges to zero as $t \to \infty$ as long as $u_0 \in L^1(R_+)$.

To deal with the second term on the left-hand side of the integral equation (18), we rewrite it in the form

$$\int_0^t G''_{t-s}(V_0(t-s))\beta(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}(ik)^2\left(\int_0^t \beta(t-s)e^{-s(k^4-ikV_0)}ds\right)dk.$$

Since $\beta \in L^\infty_{loc}(R_+)$ with the assumed limit in (22), we apply Lebesgue's Dominated Convergence Theorem and compute the integral in the principal value sense:

$$\lim_{t\to\infty}\int_0^t G''_{t-s}(V_0(t-s))\beta(s)ds = \frac{-\beta_\infty}{2\pi}\int_{-\infty}^{+\infty}\frac{k}{k^3-iV_0}dk = \frac{-\beta_\infty}{\pi}\int_0^\infty\frac{k_4 dk}{k^6+V_0^2} = \frac{-\beta_\infty}{3|V_0|^{1/3}}.$$

The first term on the left-hand side of the integral equation (12) is more tricky. First, we rewrite it in the form,

$$\int_0^t G_{t-s}(V_0(t-s))\gamma(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^t \gamma(t-s)e^{-s(k^4-ikV_0)}ds\right)dk.$$

However, if $\gamma \in L^{\infty}_{loc}(R_+)$ with the assumed limit in (22), application of Lebesgue's Dominated Convergence Theorem yields the integral in k with a simple pole at $k = 0$:

$$\lim_{t\to\infty}\int_0^t G_{t-s}(V_0(t-s))\gamma(s)ds = \frac{\gamma_\infty}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k(k^3 - iV_0)}.$$

The integral is no longer understood in the principal value sense. Instead, we return back to the treatment of the inverse Laplace transform in (11) with $Re(c) > 0$, use transformation $p = ik$, and shift the contour of integration in k below the pole at $k = 0$. As a result, computations of the integral above are completed with the half-residue term at the simple pole at $k = 0$ and the principal value integral:

$$\lim_{t\to\infty}\int_0^t G_{t-s}(V_0(t-s))\gamma(s)ds = \frac{\gamma_\infty}{2\pi}\left(\frac{\pi}{|V_0|} + \int_{-\infty}^{\infty}\frac{k^2dk}{k^6 + V_0^2}\right) = \frac{2\gamma_\infty}{3|V_0|}.$$

Combining all computations together, we obtain the following linear equation on β_∞ and γ_∞ from the integral equation (18) in the limit $t \to \infty$:

$$\frac{2\gamma_\infty}{|V_0|} - \frac{\beta_\infty}{|V_0|^{1/3}} = \frac{1}{2|V_0|^{2/3}}. \tag{23}$$

To deal with the second integral equation (20), we use the Fourier transform again to write

$$H_t(V_0t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-t(k^4-ikV_0)}}{k}dk$$

and

$$\int_0^t H_{t-s}(V_0(t-s))\beta'(s)ds = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{1}{k}\left(\int_0^t \beta'(t-s)e^{-s(k^4-ikV_0)}ds\right)dk,$$

where the integrals are understood in the principal value sense. If $\beta, \beta', \gamma \in L^{\infty}_{loc}$ with the assumed limits (22), Lebesgue's Dominated Convergence Theorem implies that

$$H_t(V_0t), \quad \int_0^t H_{t-s}(V_0(t-s))\beta'(s)ds \to 0 \quad as \quad t\to\infty.$$

Similar to the previous computations, we prove that

$$\lim_{t\to\infty}\int_0^\infty G_t'(V_0t - y)u_0(y)dy = 0,$$

$$\lim_{t\to\infty}\int_0^t G_{t-s}''(V_0(t-s))ds = \frac{-1}{3|V_0|^{1/3}},$$

$$\lim_{t\to\infty}\int_0^t G_{t-s}'(V_0(t-s))\gamma(s)ds = \frac{\gamma_\infty}{3|V_0|^{2/3}},$$

and

$$\lim_{t\to\infty}\int_0^t G_{t-s}(V_0(t-s))\beta(s)ds = \frac{\beta_\infty}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k(k^3-iV_0)} = \frac{\beta_\infty}{6|V_0|},$$

where the last integral is computed in the principal value sense because equations (19) and (20) are derived in the principal value sense.

Combining all computations together, we have obtained the following linear equation on β_∞ and γ_∞ from the integral equation (20) in the limit $t \to \infty$:

$$4\beta_\infty - \frac{2\gamma_\infty}{|V_0|^{2/3}} = \frac{1}{|V_0|^{1/3}}. \qquad (24)$$

Solving the linear system (23) and (24), we obtain (21) and the theorem is proved.

Coming back to the original question, if $u_0(0) = 0$, $u_0'(0) < 0$, and $u_0''(0) = -\frac{1}{2}$, then there is a finite value of $t_0 \in (0,\infty)$ such that $u_x(0,t) > 0$ for all $t > t_0$. Therefore, like in the case $V_0 = 0$, the function $h(x,t)$ loses monotonicity at $x = 0$ in a finite time t_0 (where $u = h_x$) with the only difference that $u_x(0,t)$ remains finite and positive as $t \to \infty$. We conclude that the presence of the advection term with $V_0 < 0$ in the fourth-order diffusion equation (6) does not prevent the loss of monotonicity in x in a finite time but still stabilizes the solution globally as $t \to \infty$. In both cases $V_0 = 0$ and $V_0 < 0$, the monotonicity of h in x is lost because of the inhomogeneous boundary condition $h_{xx}(0,t) = -\frac{1}{2}$.

3. Solution of the original problem

We shall now use Laplace transform (7) to obtain the implicit solution to the advection-diffusion equation (4) with a variable speed $V(t)$. Let us denote

$$W(t) = \int_0^t V(s)ds$$

and obtain the Laplace transform solution in the form:

$$U(p,t) = U_0(p)e^{-tp^4+W(t)p} + \int_0^t e^{-(t-s)p^4+(W(t)-W(s))p}\left(-\frac{1}{2}p + p^2\beta(s)\right)ds. \qquad (25)$$

Compared with the solution (10), we have set $\gamma(t) = 0$ because of the third boundary condition in (5). Using the inverse Laplace transform and recalling the definition of Green's function $G_t(x)$ (see Appendix A), we obtain the analogue of the implicit solution (12):

$$\begin{aligned} u(x,t) = &\int_0^\infty G_t(x+W(t)-y)u_0(y)dy - \frac{1}{2}\int_0^t G_{t-s}'(x+W(t)-W(s))ds \\ &+ \int_0^t G_{t-s}''(x+W(t)-W(s))\beta(s)ds. \end{aligned} \qquad (26)$$

Now we have two unknowns β and W and two integral equations from the boundary conditions $u(0,t) = 0$ and $u_x(0,t) = \beta(t)$.

From the boundary condition $u(0,t) = 0$, we obtain the integral equation:

$$-\int_0^t G_{t-s}(W(t)-W(s))\beta(s)ds = \int_0^\infty G_t(W(t)-y)u_0(y)dy - \frac{1}{2}\int_0^t G_{t-s}(W(t)-W(s))ds. \quad (27)$$

To find another integral equation from the boundary condition $u_x(0,t) = \beta(t)$, we differentiate the solution (26) in x:

$$\begin{aligned} u_x(x,t) &= \int_0^\infty G_t(x+W(t)-y)u_0(y)dy - \frac{1}{2}\int_0^t G''_{t-s}(x+W(t)-W(s))ds \\ &+ \frac{1}{2}\beta(t) - \beta(0)H_t(x+W(t)) - \int_0^t H_{t-s}(x+W(t)-W(s))\beta'(s)ds \\ &+ V(t)\int_0^t G_{t-s}(x+W(t)-W(s))\beta(s)ds. \end{aligned} \quad (28)$$

From the boundary condition $u_x(0,t) = \beta(t)$, we obtain another integral equation:

$$\begin{aligned} &\beta(t) + 2\beta(0)H_t(W(t)) + 2\int_0^t H_{t-s}(W(t)-W(s))\beta'(s)ds - 2V(t)\int_0^t G_{t-s}(W(t)-W(s))\beta(s)ds \\ &= 2\int_0^\infty G'_t(W(t)-y)u_0(y)dy - \int_0^t G''_{t-s}(W(t)-W(s))ds. \end{aligned} \quad (29)$$

We shall prove that the system of two integral equations (27) and (29) determines uniquely the function $\beta(t)$ and $V(t)$ locally for $t > 0$. The following theorem gives the result in the form of the fractional power series in t.

Theorem 3. *Assume that* $u_0 \in C^\infty(R_+)$ *such that*

$$u_0(0) = 0, \quad u''_0(0) = -\frac{1}{2}, \quad u'''_0(0) = 0. \quad (30)$$

Then, there exists a formal solution (V, β) of the system of two integral equations (27) and (29) in the form of the fractional power series:

$$\beta(t) = \beta_0 + \sum_{n=4}^\infty \beta_{n/4}t^{n/4}, \quad V(t) = V_0 + \sum_{n=1}^\infty V_{n/4}t^{n/4}, \quad (31)$$

where $\beta_0 = u'_0(0)$, $V_0 = u_0^{(4)}(0)/u'_0(0)$, *and* $\{\beta_{n/4}, V_{(n-3)/4}\}_{n=4}^\infty$ *are uniquely determined.*

Proof. We substitute the series representations (31) to each term of the integral equations (27) and (29). It follows from (31) that

$$a_t := \frac{1}{t^{1/4}}\int_0^t V(s)ds = V_0 t^{3/4} + \sum_{n=1}^\infty \frac{4}{n+4}V_{n/4}t^{(n+3)/4}$$

and

$$\xi_{t,\tau} := \frac{1}{\tau^{1/4}}\int_{t-\tau}^t V(s)ds = V_0\tau^{3/4} + \sum_{n=1}^\infty \frac{4}{n+4}V_{n/4}\frac{t^{(n+4)/4} - (t-\tau)^{(n+4)/4}}{\tau^{1/4}}.$$

Using the representation (A.5) of the Green function with $g \in C^\infty(R)$, we obtain for the three terms of the integral equation (27):

$$\begin{aligned} \int_0^t G''_{t-s}(W(t)-W(s))\beta(s)ds &= \beta_0\int_0^t \frac{g''(\xi_{t,\tau})}{\tau^{3/4}}d\tau + \sum_{n=4}^\infty \beta_{n/4}\int_o^t \frac{g''(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{3/4}}d\tau \\ &= 4\beta_0 g''(0)t^{1/4} + \sum_{k=2}^\infty \frac{1}{k!}g^{(k+2)}(0)\int_0^t \frac{\xi_{t,\tau}^k}{\tau^{3/4}}d\tau + \sum_{n=4}^\infty \beta_{n/4}\int_0^t \frac{g''(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{3/4}}d\tau, \end{aligned}$$

$$\int_0^\infty G_t(W(t)-y)u_0(y)dy=\int_0^\infty g(z-a_t)u_0(t^{1/4}z)dz=\sum_{n=1}^\infty\frac{1}{n!}u_0^{(n)}(0)t^{n/4}\int_0^\infty g(z-a_t)z^n dz$$

$$=t^{1/4}u_0(0)\sum_{k=0}^\infty\frac{1}{k!}(-a_t)^k\int_0^\infty g^{(k)}(z)zdz+\sum_{n=2}^\infty\frac{1}{n!}u_0^{(n)}(0)t^{n/4}\int_0^\infty g(z-a_t)z^n dz,$$

and

$$\int_0^t G'_{t-s}(W(t)-W(s))ds=\int_0^t\frac{g'(\xi_{t,\tau})}{\tau^{2/4}}d\tau=\sum_{k=1}^\infty\frac{1}{k!}g^{(k+1)}(0)\int_0^t\frac{\xi_{t,\tau}^k}{\tau^{2/4}}d\tau.$$

At the first powers of $t^{1/4}$, we obtain a system of linear algebraic equations on the coefficients of the fractional power series (31):

$$t^{1/4}:\qquad -4\beta_0 g''(0)=u_0'(0)\int_0^\infty g(z)zdz,$$

$$t^{2/4}:\qquad 0=\frac{1}{2!}u_0''(0)\int_0^\infty g(z)z^2dz,$$

$$t^{3/4}:\qquad 0=\frac{1}{3!}u_0'''(0)\int_0^\infty g(z)z^3dz,$$

$$t^{4/4}:\qquad 0=\frac{1}{4!}u_0^{(4)}(0)\int_0^\infty g(z)z^4dz-u_0'(0)V_0\int_0^\infty g'(z)zdz,$$

$$t^{5/4}:\qquad -\beta_{4/4}g''(0)\int_0^1\frac{(1-x)^{4/4}}{x^{3/4}}dx=\frac{1}{5!}u_0^{(5)}(0)\int_0^\infty g(z)z^5dz-\frac{1}{2}u_0''(0)V_0\int_0^\infty g'(z)z^2dz$$

$$-\frac{4}{5}u_0'(0)V_{1/4}\int_0^\infty g'(z)zdz-\frac{2}{5}g''(0)V_0,$$

and so on.

Using the explicit values for the integrals (A.9)–(A.13) and the initial conditions (30), we obtain $\beta_0=u_0'(0),\ V_0=u_0^{(4)}(0)/u_0'(0)$, and the linear equation

$$u_0'(0)V_{1/4}+8g''(0)\left(\beta_{4/4}+u_0^{(5)}(0)+\frac{1}{2}V_0\right)=0. \qquad (32)$$

Similarly, we expand all terms of the second integral equation (29):

$$\int_0^t G_{t-s}(W(t)-W(s))\beta(s)ds=\beta_0\int_0^t\frac{g(\xi_{t,\tau})}{\tau^{1/4}}d\tau+\sum_{n=4}^\infty\beta_{n/4}\int_0^t\frac{g(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{1/4}}d\tau$$

$$=\frac{4}{3}\beta_0 g(0)t^{3/4}+\sum_{k=2}^\infty\frac{1}{k!}g^{(k)}(0)\int_0^t\frac{\xi_{t,\tau}^k}{\tau^{1/4}}d\tau+\sum_{n=4}^\infty\beta_{n/4}\int_0^t\frac{g(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{1/4}}d\tau,$$

$$\int_0^\infty G_t'(W(t)-y)u_0(y)dy=\int_0^\infty g(z-a_t)u_0'(t^{1/4}z)dz=\sum_{n=0}^\infty\frac{1}{n!}u_0^{(n+1)}(0)t^{n/4}\int_0^\infty g(z-a_t)z^n dz$$

$$=u_0'(0)\sum_{k=0}^\infty\frac{1}{k!}(-a_t)^k\int_0^\infty g^{(k)}(z)dz+\sum_{n=1}^\infty\frac{1}{n!}u_0^{(n+1)}(0)t^{n/4}\int_0^\infty g(z-a_t)z^n dz,$$

$$\int_0^t G''_{t-s}(W(t)-W(s))ds=\int_0^t\frac{g''(\xi_{t,\tau})}{\tau^{3/4}}d\tau=\sum_{k=0}^\infty\frac{1}{k!}g^{(k+2)}(0)\int_0^t\frac{\xi_{t,\tau}^k}{\tau^{3/4}}d\tau,$$

$$H_t(W(t)) = \int_0^{a_t} g(z)dz = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} g^{(k)}(0) a_t^{k+1},$$

and

$$\int_0^t H_{t-s}(W(t)-W(s))\beta'(s)ds = \sum_{n=4}^{\infty} \beta_{n/4} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} g^{(k)}(0) \int_0^t \xi_{t,\tau}^{k+1} (t-\tau)^{(n-4)/4} d\tau.$$

At the first powers of $t^{1/4}$, we obtain a system of linear algebraic equations on the coefficients of the fractional power series (31):

$$t^{0/4}: \qquad \beta_0 = 2u_0'(0)\int_0^{\infty} g(z)dz,$$

$$t^{1/4}: \qquad 0 = 2u_0''(0)\int_0^{\infty} g(z)zdz - 4g''(0),$$

$$t^{2/4}: \qquad 0 = u_0'''(0)\int_0^{\infty} g(z)z^2dz,$$

$$t^{3/4}: \qquad 2\beta_0 g(0)V_0 - \frac{8}{3}\beta_0 g(0)V_0 = \frac{1}{3}u_0^{(4)}(0)\int_0^{\infty} g(z)z^3dz - 2u_0'(0)V_0\int_0^{\infty} g'(z)dz,$$

$$t^{4/4}: \qquad \beta_{4/4} + \frac{8}{5}\beta_0 g(0)V_{1/4} - \frac{8}{3}\beta_0 g(0)V_{1/4} = \frac{1}{12}u_0^{(5)}(0)\int_0^{\infty} g(z)z^4dz - 2u_0''(0)V_0\int_0^{\infty} g'(z)zdz - \frac{8}{5}u_0'(0)V_{1/4}\int_0^{\infty} g'(z)dz,$$

and so on. Again, using the explicit values for the integrals (A9)–(A.13) and the initial conditions (30), we obtain $\beta_0 = u_0'(0),\ V_0 = u_0^{(4)}(0)/u_0'(0)$, and the linear equation

$$-\frac{8}{3}u_0'(0)g(0)V_{1/4} + (\beta_{4/4} + u_0^{(5)}(0) + \frac{1}{2}V_0) = 0. \tag{33}$$

The system of linear equations (32) and (33) has a unique solution

$$V_{1/4} = 0, \quad \beta_{4/4} = -u_0^{(5)}(0) - \frac{1}{2}V_0, \tag{34}$$

provided that

$$-\frac{64}{3}g(0)g''(0) = \frac{4}{3\pi^2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{4\sqrt{2}}{3\pi} \neq 1,$$

which is true. Note that the constraint $V_0 = u_0^{(4)}(0)/u_0'(0)$ also follows from the pointwise equation (2) obtained for sufficiently smooth solutions. Similarly, the second equation (34) follows from the advection-diffusion equation (4) after one derivative in x and the limit $x \to 0$ and $t \to 0$.

It remains to prove that the system of linear equations obtained from the system of integral equations (27) and (29) can be solved at each order of $t^{(n+1)/4}$ and $t^{n/4}$, respectively, for $n \geq 4$. From the previous computations, we can deduce that the first integral equation at $t^{(n+1)/4}$ gives a linear equation on variables ($\beta_{n/4}, V_{(n-3)/4}$) of the power series (31):

$$-\beta_{n/4} g''(0) \int_0^1 \frac{(1-x)^{n/4}}{x^{3/4}} dx + \frac{4}{n+1} u_0'(0) V_{(n-3)/4} \int_0^\infty g'(z) z dz = \cdots, \qquad (35)$$

where the dots on the right-hand side denote the terms expressed through derivatives of $u_0(x)$ at $x = 0$ and the previous terms of the power series (31). Similarly, the second integral equation at $t^{n/4}$ gives another linear equation on variables ($\beta_{n/4}, V_{(n-3)/4}$):

$$\beta_{n/4} - \frac{8}{3}\beta_0 g(0) V_{(n-3)/4} = \cdots. \qquad (36)$$

The system of linear equations (35) and (36) is non-degenerate if

$$C_n := -\frac{4(n+1)}{3} g(0) g''(0) \int_0^1 \frac{(1-x)^{n/4}}{x^{3/4}} dx = \frac{(n+1)}{6\sqrt{2}\pi} \frac{\Gamma\left(\frac{n+4}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{n+5}{4}\right)} \neq 1. \qquad (37)$$

The coefficients $\{C_n\}$ are computed numerically for $n \geq 1$ (see Fig. 2). The sequence is monotonically increasing. It approaches closely to 1 at $n = 8$, where $C_8 \approx 0.96$, and $n = 9$, where $C_9 \approx 1.04$. Therefore, $C_n \neq 1$ for all $n \geq 1$ so that the linear system is non-degenerate and a unique solution for $(\beta_{n/4}, V_{(n-3)/4})$ exists for any $n \geq 4$.

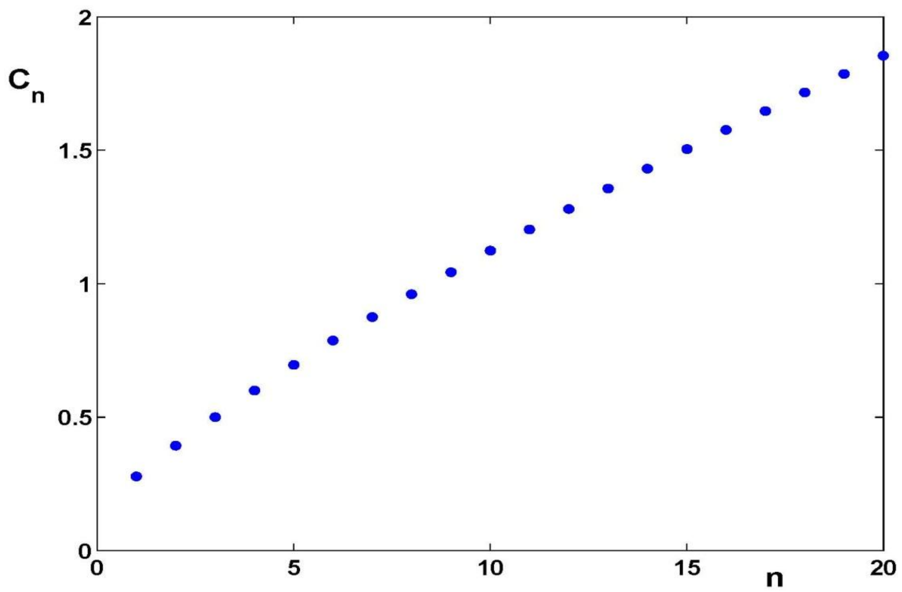

Fig. 2. Numerical approximations of C_n defined by (37)

In the present time, we cannot prove yet that the system of integral equations (27) and (29) leads to a finite-time blow-up, according to the conjecture in [1]. Nevertheless, numerical computations show that the blow-up holds for a generic set of initial data. Fig. 3 shows the behavior of functions $\beta(t)$ and $V(t)$ near the blow-up time. It follows from this figure that $\beta(t) = h_{xx}(0,t) \to 0$ at the same time as $V(t) \to -\infty$ with $\beta(t)\ V(t)^{1/3} \to C_0$, where $C_0 > 0$ is a numerical constant. In other words, we conclude with the conjecture that $\beta(t) \sim V(t)^{-1/3}$ as $V(t) \to -\infty$ in a finite time $t_0 \in (0,\infty)$.

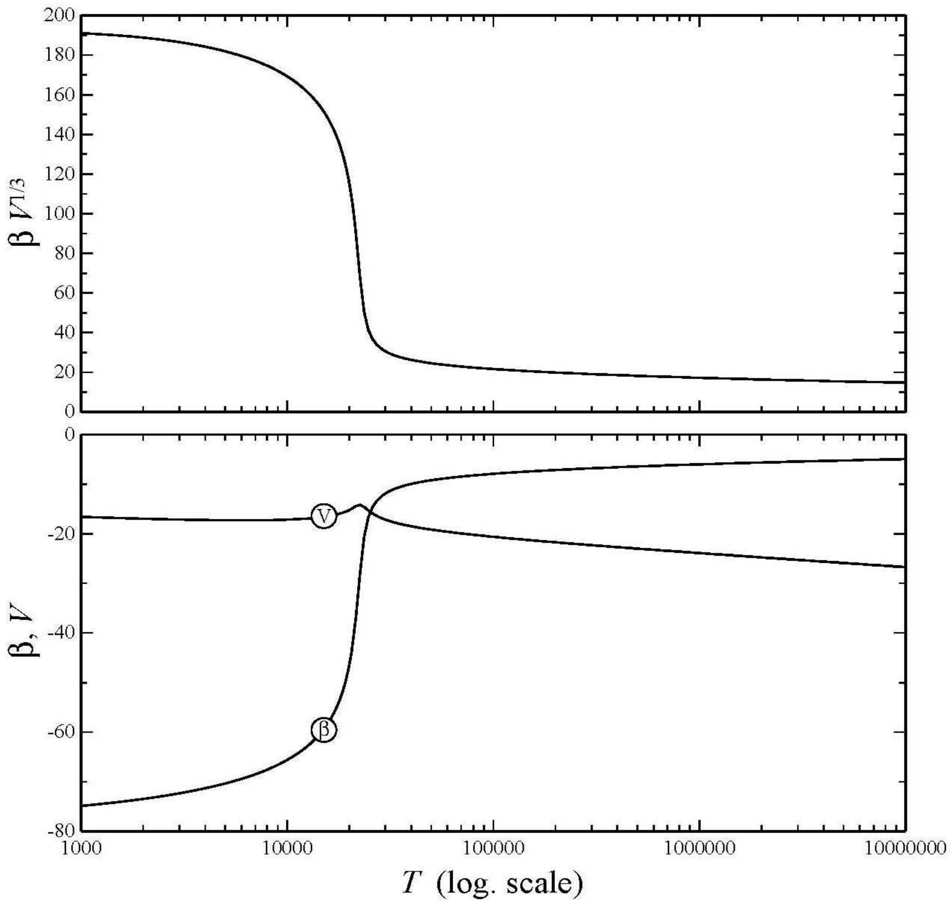

Fig. 3. Numerical computations of β(*t*) and *V*(*t*) for the advection-diffusion equation (1).

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Appendix. Green's function

Let us define the fundamental solution of the fourth-order diffusion equation:

$$\begin{cases} h_t + h_{xxxx} = 0, & x \in R, \quad t > 0, \\ h|_{t=0} = \delta(x), & x \in R, \end{cases} \tag{A.1}$$

where δ is a standard Dirac delta-function in the distribution sense.The fundamental solution is usually referred to as Green's function and we shall denote it by

$$h(x,t) = G_t(x), \quad x \in R, \quad t \in R_+.$$

Using the Fourier transform in *x*, we can obtain the explicit expression for Green's function:

$$G_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tk^4+ikx}dk = \frac{1}{\pi}\int_0^{\infty} e^{-tk^4}\cos(kx)dk. \tag{A.2}$$

In particular, we have $G_t(-x) = G_t(x)$ for all $x \in R$ and

$$G_t(0) = \frac{1}{\pi}\int_0^{\infty} e^{-tk^4}dk = \frac{\Gamma(1/4)}{4\pi t^{1/4}}, \tag{A.3}$$

$$G_t''(0) = -\frac{1}{\pi}\int_0^\infty k^2 e^{-tk^4}\,dk = -\frac{\Gamma(3/4)}{4\pi t^{3/4}}, \tag{A.4}$$

where Γ is the standard Gamma function.The Green's function can be represented in the self-similar form by

$$G_t(x) = \frac{1}{t^{1/4}}\, g\left(\frac{x}{t^{1/4}}\right), \quad g(z) = \frac{1}{\pi}\int_0^\infty e^{-k^4}\cos(kz)dk, \tag{A.5}$$

where $g \in L^2(R) \cap L^\infty(R)$. Therefore, G_t decays to zero as $t \to \infty$ in any L^p norm for $p \geq 2$. In particular, $|G_t(x)| \leq \|g\|_{L^\infty}/t^{1/4}$, $|G_t'(x)| \leq \|g'\|_{L^\infty}/t^{1/2}$, and so on, for any $x \in R$.

By the stationary phase method (see, e.g., Chapter 5 in [3]), $g(z)$ and all derivatives of $g(z)$ decay to zero as $|z| \to \infty$ faster than any algebraic powers. This gives the decay of $G_t(x)$ and any x-derivative of $G_t(x)$ as $|x| \to \infty$ for any fixed $t > 0$. Although G_t and g are not L^1 functions, they satisfy the normalization conditions:

$$\int_R G_t(x)dx = \int_R g(z)dz = 1, \quad t > 0. \tag{A.6}$$

The even function g: $R \to R$ satisfies the ordinary differential equation

$$4\frac{d^4 g}{dz^4} = g + z\frac{dg}{dz}, \quad z \in R, \tag{A.7}$$

subject to the initial values

$$g(0) = \frac{1}{4\pi}\Gamma\left(\frac{1}{4}\right), \quad g'(0) = 0, \quad g''(0) = -\frac{1}{4\pi}\Gamma\left(\frac{3}{4}\right), \quad g'''(0) = 0, \tag{A.8}$$

and the decay behavior as $|z| \to \infty$. It is clear from the differential equation that $g \in C^\infty(R)$ satisfies a number of integral constraints:

$$\int_0^\infty z g(z)dz = -4g''(0), \tag{A.9}$$

$$\int_0^\infty z^2 g(z)dz = 0, \tag{A.10}$$

$$\int_0^\infty z^3 g(z)dz = -8g(0), \tag{A.11}$$

$$\int_0^\infty z^4 g(z)dz = -12, \tag{A.12}$$

$$\int_0^\infty z^5 g(z)dz = 164!\,g''(0), \tag{A.13}$$

and so on.

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О РЕШЕНИЯХ УПРОЩЕННОЙ МОДЕЛИ ДИНАМИЧЕСКОЙ ЭВОЛЮЦИИ КОНТАКТНЫХ ЛИНИЙ

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Цель: Цель работы заключается в решении линейного уравнения адвекции-диффузии с переменной скоростью на полубесконечной оси. Переменная скорость определяется дополнительным условием на границе, которое моделирует динамику контактной линии гидродинамического потока с контактным углом 180˚.
Научный подход: Исследование проведено с применение преобразования Лапласа по пространственной координате. Свойства функции Грина для уравнения диффузии 4-го порядка использованы при анализе неявных решений линейного уравнения адвекции-диффузии.
Результат: Доказано локальное существование решения начальной задачи ассоциированной с набором переопределенных граничных условий в форме дробно-степенного ряда по временной переменной. Проанализированы явные решения в случае постоянной скорости, что показывает, что неоднородные граничные условия приводят к изменению выпуклости потока в контактной линии за конечное время.

Ключевые слова: линейное уравнение адвекции-диффузии, переменная скорость, контактная линия, преобразование Лапласа, функция Грина.